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Rigidity, functional equations and the Calogero–Moser model

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Abstract

Suppose we have a natural Hamiltonian H of n particles on the line, centre-of-mass momentum P and a further independent quantity Q, cubic in the momenta. If these are each S_n invariant and mutually Poisson commute we have the Calogero–Moser system with potential $V = \frac{1}{6}\sum_{i\neq j}\wp(q_i - q_j) + \text{const}$.

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1. Introduction

The following paper deals with many-particle Hamiltonian systems on the line and their integrability. Although such systems arise in many physical settings and have been extensively studied there is still no simple way to determine their integrability or otherwise. General arguments [17] tell us that many-particle Hamiltonian systems for sufficiently repulsive potentials are integrable, yet there appear to be few direct methods of actually solving for such systems. The integrable systems we can actually solve seem to form a very privileged class. The result presented here sheds some light on this state of affairs. We will follow a less well known route to the study of integrable systems, that employing functional equations.

Here we address the following question: what S_n-invariant, natural Hamiltonian systems of n particles on the line and conserved centre-of-mass momentum admit a third independent, S_n invariant, mutually Poisson commuting quantity, cubic in the momenta? (The precise statement and explanation of these terms will be given below.) Our answer is somewhat surprising. These data characterize the a_n Calogero–Moser systems. This is the 'rigidity' of our title. Although no restrictions were placed on further Poisson commuting invariants we arrive at a system for which sufficient exist to yield complete integrability. The mixture of symmetry and polynomial momentum is powerful. Such natural requirements and our result go some way in explaining the ubiquity of this class of models. The situation is somewhat reminiscent of the original work of Ruijsenaars and Schneider [28] who, when demanding certain commutation properties, discovered a class of Hamiltonian systems that proved to be integrable. It is also analogous to what one encounters with W and related algebras, where a few commutation relations specify the whole structure. Indeed, the connections between conformal field theory and these models may mean this is more than analogy [18, 23]. In the quantum regime analogous and stronger properties are already known (and will be remarked upon further in due course) when the symmetry and polynomial dependence above are replaced by symmetry and holomorphicity. Again, a few commutation relations specify the whole structure and the result we obtain here is the classical version of a result of [27, proposition 4.2 and remark 4.4].

There are obvious generalizations to this paper which will be taken up in the discussion. Before turning to the statement and proof of the result (given in the following two sections) it is perhaps worth making some general remarks on connections between integrable systems and functional equations. Functional equations have, of course, a long and interesting history in connection with mathematical physics and touch upon many branches of mathematics [1, 2]. They have arisen in the context of completely integrable systems in several different ways. We have already mentioned the work of Ruijsenaars and Schneider [28]. Hietarinta [19] similarly derived a functional equation when seeking a second quartic integral for two-particle systems on the line. A further way in which they arise is by assuming an ansatz for a Lax pair, the consistency of the Lax pair yielding functional and algebraic constraints. In this manner Calogero discovered the elliptic Calogero–Moser model [13] and Bruschi and Calogero [8, 9] constructed Lax pairs for the Ruijsenaars models. The functional equations found by this route appear [4] as particular examples of

$$\phi_1(x+y) = \frac{\begin{vmatrix} \phi_2(x) & \phi_2(y) \\ \phi_3(x) & \phi_3(y) \end{vmatrix}}{\begin{vmatrix} \phi_4(x) & \phi_4(y) \\ \phi_5(x) & \phi_5(y) \end{vmatrix}}.$$

The general analytic solution of this has been given by Braden and Buchstaber [5]. Interestingly, Novikov's school have shown that the Hirzebruch genera associated with the index theorems of known elliptic operators arise as solutions of functional equations which are particular examples of this. The string-inspired Witten index was shown by Ochanine to be described by Hirzebruch's construction where now the elliptic function solutions were important [20]. A similar approach based upon an ansatz and consequent functional equations was used by Inozemtsev [21] to construct generalizations of the Calogero–Moser models, while in [6] this route was used to construct new solutions to the WDVV equations. Various functional equations have also arisen when studying the properties of wavefunctions for associated quantum integrable problems. Gutkin found several functional relations by requiring a nondiffractive potential [16] while Calogero [14] and Sutherland [29, 30] obtained functional relations by seeking factorizable ground-state wavefunctions. A recurring equation in this latter approach is

$$\begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(y) & h(z) \\ f'(x) & g'(y) & h'(z) \end{vmatrix} = 0 \qquad x+y+z=0.$$

This finds a general solution in [3, 10]. In this paper we will make use of the particular case of this equation (variously solved under evenness and holomorphicity constraints in [27], analyticity constraints in [10], and most generally in [3]).

Theorem 1. *Let f be a three-times differentiable function satisfying the functional equation*

$$\begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ f'(x) & f'(y) & f'(z) \end{vmatrix} = 0 \qquad x+y+z=0. \tag{1}$$

Up to the manifest invariance

$$f(x) \to \alpha f(\delta x) + \beta$$

the solutions of (1) are one of $f(x) = \wp(x+d)$, $f(x) = \mathrm{e}^x$ *or* $f(x) = x$. *Here* $\wp$ *is the Weierstrass* $\wp$ *function and* $3d$ *is a lattice point of the* $\wp$ *function.*

Perhaps one reason for the underlying connection between integrability and functional equations is the fact that Baker–Akhiezer functions obey such relations. Such connections between integrable functional equations and algebraic geometry have been studied by Buchstaber and Krichever [11] and Dubrovin *et al* [15]. Whatever, these connections between functional equations and complete integrability warrant further investigation.

2. The result

The result discovered is the following.

Theorem 2. *Let* H *and* P *be the (natural) Hamiltonian and centre-of-mass momentum:*

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V \qquad P = \sum_{i=1}^{n} p_i. \tag{2}$$

Denote by Q *an independent third-order quantity:*

$$Q = \sum_{i=1}^{n} p_i^3 + \frac{1}{6}\sum_{i\neq j\neq k} d_{ijk} p_i p_j p_k + \sum_{i\neq j} d_{ij} p_i^2 p_j + \frac{1}{2}\sum_{ij} a_{ij} p_i p_j + \sum_i b_i p_i + c. \tag{3}$$

If these are S_n*-invariant and Poisson commute,*

$$\{P, H\} = \{P, Q\} = \{Q, H\} = 0$$

then $V = \frac{1}{6}\sum_{i\neq j}\wp(q_i - q_j) + \text{const}$ *and we have the Calogero–Moser system.*

Some explanatory remarks are in order. Here S_n invariance means that for any coefficient $\alpha_{ij}(q_1, q_2, \ldots, q_n)$ in the expansions above we have $\alpha_{\sigma(i)\sigma(j)}(q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(n)})$ for all $\sigma \in S_n$. In particular $V(q_1, q_2, \ldots, q_n) = V(q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(n)})$ for all $\sigma \in S_n$. We remark that, had we begun with particles of possibly different particle masses, $H = \frac{1}{2}\sum_{i=1}^{n} m_i p_i^2 + V$, the effect of S_n invariance is such as to require these masses to be the same. Thus we are assuming the S_n-invariant Hamiltonian of the introduction. Finally, by 'an independent third-order quantity' Q we mean one functionally independent of H and P and for which one cannot obtain an invariant of lower degree by subtracting multiples of P^3 and PH. We are not dealing with quadratic conserved quantities here.

3. The proof

Our proof has five steps. We begin by noting that the Poisson commutativity $\{Q, H\} = 0$ yields (with $\{q_i, p_j\} = \delta_{ij}$)

$$\begin{aligned} 0 = {} & \frac{1}{6}\sum_l \sum_{i\neq j\neq k} (\partial_l d_{ijk}) p_i p_j p_k p_l + \sum_l \sum_{i\neq j} (\partial_l d_{ij}) p_i^2 p_j p_l + \sum_{i,j,l} (\partial_l a_{ij}) p_i p_j p_l \\ & + \sum_{i,j} (\partial_i b_j) p_i p_j - 3\sum_i p_i^2 (\partial_i V) - \frac{1}{2}\sum_{i\neq j\neq k} d_{ijk} (\partial_k V) p_i p_j \\ & - \sum_{i\neq j} d_{ij} (2(\partial_i V) p_i p_j + (\partial_j V) p_i^2) \\ & - \sum_{i,j} a_{ij} (\partial_i V) p_j + \sum_i (\partial_i c) p_i - \sum_i b_i \partial_i V. \end{aligned} \tag{4}$$

The steps then are:

(1) First we show that the d_{ijk} and d_{ij} terms in (3) may be taken to be zero.
Having made this simplification we then focus on the terms remaining in (4) independent and quadratic in the momenta:

$$\partial_j b_i + \partial_i b_j = 0 \qquad i \neq j \tag{5}$$
$$\partial_i b_i - 3\partial_i V = 0 \tag{6}$$
$$\sum b_i \partial_i V = 0. \tag{7}$$

(2) Second, using (5), (6) we show that b_j may be written in the form

$$b_j = \sum_{i \neq j} W(q_i - q_j) + U(q_j) \tag{8}$$

where W is an even function.
(3) Third, using $\{P, Q\} = 0$, we may set $U = 0$.
(4) Fourth, that we may rewrite (7) in the form

$$0 = \sum_{i<j<k} \begin{vmatrix} 1 & 1 & 1 \\ W(q_i - q_j) & W(q_j - q_k) & W(q_k - q_i) \\ W'(q_i - q_j) & W'(q_j - q_k) & W'(q_k - q_i) \end{vmatrix}. \tag{9}$$

(5) Finally we argue that each term in the sum (9) itself vanishes and so we arrive at an equation of the form (1). The result then follows simply.

Step 1. We begin by focusing on the terms in (4) quartic in the momenta. For l different from i, j, k we see that $\partial_l d_{ijk} = 0$, and so $d_{ijk} = d_{ijk}(q_i, q_j, q_k)$. Further, from the coefficients of $p_i^3 p_j$, $p_i^2 p_j^2$ and $p_i^2 p_j p_k$ (for i, j, k distinct), respectively, we find

$$\partial_i d_{ij} = 0 \qquad \partial_j d_{ij} = 0 \qquad \partial_j d_{ik} + \partial_k d_{ij} + \partial_i d_{ijk} = 0. \tag{10}$$

The first and third of these together show $\partial_i^2 d_{ijk} = 0$ and so d_{ijk} is at most linear in q_i. By symmetry

$$d_{ijk} = \alpha q_i q_j q_k + \beta(q_i q_j + q_j q_k + q_k q_i) + \gamma(q_i + q_j + q_k) + \delta.$$

Now, using $\{P, Q\} = 0$ shows $\alpha = \beta = \gamma = 0$. Thus d_{ijk} is a constant. This fact, together with the second and third equations of (10), shows $\partial_k^2 d_{ij} = 0$. Therefore d_{ij} is at most linear in q_k (for $k \neq i, j$). The first two equations of (10) show d_{ij} is independent of q_i and q_j. Now a similar argument employing $\{P, Q\} = 0$ yields d_{ij} to be constant. By subtracting appropriate multiples of P^3 and PH we may then remove the d terms from Q. Our assumption of independence means that the leading term of Q does not vanish when doing this. Thus (after such a subtraction and a possible rescaling) we may set the d terms in Q to be zero. Henceforth we will assume this simplification has been made.

Step 2. Suppose i, j, k are distinct. Then from (5) we obtain ($\partial_{ij} = \partial_i \partial_j$, etc)

$$\partial_{jk} b_i + \partial_{ik} b_j = 0 \qquad \partial_{jk} b_i + \partial_{ij} b_k = 0.$$

Taking the difference of these we see $\partial_i(\partial_k b_j - \partial_j b_k) = 0$ and so

$$-\partial_k b_j + \partial_j b_k = 2F(q_j, q_k).$$

Combining this with $\partial_k b_j + \partial_j b_k = 0$ we obtain

$$\partial_j b_k = F(q_j, q_k) = -F(q_k, q_j) = -\partial_k b_j.$$

We wish to further restrict the form of F. If we apply ∂_i to (5) and then use (6) we see

$$-\partial_i\partial_i b_j = \partial_i\partial_j b_i = 3\partial_i\partial_j V = \partial_j\partial_i b_j$$

and so

$$(\partial_i + \partial_j)\partial_i b_j = 0.$$

Therefore

$$\partial_i b_j = F(q_i - q_j) \qquad F(x) = -F(-x). \tag{11}$$

Upon integrating we obtain (8) where $W'(x) = F(x)$ and W is an even function. (In principle, upon integrating the odd function F we obtain a function $\tilde{W}$ where $\tilde{W}'(x) = F(x)$ and $\tilde{W}(x) = \tilde{W}(-x) + \tilde{c}$. *A priori* we cannot argue that the integration constant $\tilde{c}$ vanishes if $\tilde{W}(0)$ is not defined, as happens for singular potentials. However, setting $W(x) = \frac{1}{2}(\tilde{W}(x) + \tilde{W}(-x))$ again yields (8) up to a constant, which at this stage may be incorporated into the arbitrary function U.) We have employed the S_n symmetry throughout this step to identify each of the possibly different functions F, W and U arising from each pair as the same.

Step 3. Now, upon employing $\{Q, P\} = 0$ we see $\sum_{i=1}^{n} \partial_i b_j = 0$. Using (8) we deduce that $\partial_i U(q_i) = 0$ and so $U(q_i)$ is a constant. Such a constant may be removed altogether by subtracting an appropriate multiple of P from Q, or simply incorporated into a redefinition of $W(x)$. Whatever, we may take $U = 0$. Then

$$b_i^2 = \sum_{j\neq i} W^2(q_j - q_i) + 2\sum_{j\neq k\neq i} W(q_j - q_i)W(q_k - q_i). \tag{12}$$

Step 4. Now, employing (6) and (7) we see $0 = \sum_i \partial_i b_i^2$. Using (12) we obtain

$$\partial_i b_i^2 = -2\sum_{j\neq i} W(q_j - q_i)F(q_j - q_i) + 2\sum_{j\neq k\neq i} \partial_i(W(q_j - q_i)W(q_k - q_i)).$$

When we sum this expression over i the first term will vanish using oddness and evenness properties. Thus we arrive at

$$0 = \sum_{i\neq j\neq k} \partial_i(W(q_j - q_i)W(q_k - q_i)).$$

Define A_{ijk} by

$$A_{ijk} = \partial_i(W_{ji}W_{ki}) + \partial_j(W_{ij}W_{kj}) + \partial_k(W_{ik}W_{jk}) = \begin{vmatrix} 1 & 1 & 1 \\ W_{ij} & W_{jk} & W_{ki} \\ F_{ij} & F_{jk} & F_{ki} \end{vmatrix}$$

where we use the shorthand $W_{ij} = W(q_i - q_j)$. Then from the functional form of W we know

$$A_{ijk} = A_{jik} = A_{jki} = \Psi(q_i - q_j, q_j - q_k, q_k - q_i) \tag{13}$$

and is fully symmetric in i, j, k. Thus

$$0 = \sum_{i<j<k} A_{ijk} = \sum_{i<j<k} \begin{vmatrix} 1 & 1 & 1 \\ W(q_i - q_j) & W(q_j - q_k) & W(q_k - q_i) \\ W'(q_i - q_j) & W'(q_j - q_k) & W'(q_k - q_i) \end{vmatrix} \tag{14}$$

which is equation (9).

Step 5. We now wish to argue that $A_{ijk} = 0$. If we apply ∂_{ijk} to (14) we find that

$$\partial_{ijk} A_{ijk} = 0$$

as only one term in the sum depends on i, j, k. Thus $\partial_{jk} A_{ijk}$ is independent of q_i, and consequently, due to the functional form (13), it must be a function of $q_j - q_k$ only. Therefore we must have

$$\partial_{jk} A_{ijk} = B(q_j - q_k)$$

and so, after integration and use of symmetry,

$$A_{ijk} = E(q_i - q_j) + E(q_j - q_k) + E(q_k - q_i)$$

(where $E(x) = E(-x)$ and $E''(x) = -B(x)$). We may therefore rewrite (14) as

$$0 = \sum_{i<j} E(q_i - q_j). \tag{15}$$

Taking the partial derivative ∂_{ij} of this expression then gives $\partial_{ij} E(q_i - q_j) = 0$, as only this term depends on both i and j. This, together with the evenness of E, tells us that E is a constant. In conjunction with (15) we deduce $E = 0$. That is, $A_{ijk} = 0$. Therefore for each distinct triple i, j, k

$$0 = \begin{vmatrix} 1 & 1 & 1 \\ W(q_i - q_j) & W(q_j - q_k) & W(q_k - q_i) \\ W'(q_i - q_j) & W'(q_j - q_k) & W'(q_k - q_i) \end{vmatrix}.$$

But this is none other than (1). The even solution of this is $W(x) = \wp(x)$, up to a constant. Finally, using (6) and (8) we obtain the stated conclusion.

4. Discussion

Our result may be interpreted as a rigidity theorem for the a_n Calogero–Moser system and in part explains this model's ubiquity: demanding a cubic invariant together with S_n invariance necessitates the model. A detailed scrutiny of our proof shows several generalizations possible. A natural generalization is to replace the S_n invariance with the invariance of a general Weyl group W and make a connection with the Calogero–Moser models associated with other root systems [24,25]. Quite a bit is known about the quantum generalizations in this regard. Given a commutative ring $\mathcal{R}$ of W-invariant, holomorphic, differential operators, whose highest-order terms generate W-invariant differential operators with constant coefficients, then the potential term for the Laplacian $\mathcal{H}$ (the quantum Hamiltonian) has Calogero–Moser potential appropriate to W [26,27]. In this setting it is known that the commutativity of just a few low-order elements of $\mathcal{R}$ dictate the form of the potential and the commuting algebra (at least for the classical root systems [27]). In particular, the result derived above is the classical analogue of a result of [27] for the a_n root system, where a functional equation equivalent to (9) was obtained by requiring the commutativity of certain linear, quadratic and cubic holomorphic differential operators. Taniguchi's results [31] are also indicative of the rigidity of these quantum models: if $\mathcal{H}$ is the quantum Hamiltonian just discussed, and $\mathcal{Q}_{1,2}$ are holomorphic (but not *a priori* W-invariant) differential operators of appropriate degrees for which $[\mathcal{Q}_{1,2}, \mathcal{H}] = 0$, then $\mathcal{Q}_{1,2} \in \mathcal{R}$ and consequently $[\mathcal{Q}_1, \mathcal{Q}_2] = 0$. Interestingly in this paper we have employed a functional equation encountered elsewhere in the quantum regime.

A further generalization of this paper would be to replace the natural Hamiltonian structure of our theorem with (say) Hamiltonians of Ruijsenaars type. We remark in passing that there are still several unsolved functional equations surrounding this model. One might also seek to relax the full S_n invariance imposed here. By so doing this will allow the Toda models. As shown by Inozemtsev [22], the Toda models arise as scaling limits of the Calogero–Moser model, the latter being the 'generic' situation [4]. It would be interesting to understand this in terms of the coadjoint descriptions of these models.

Though perhaps not obvious, this paper arose from trying to understand models conjectured to be integrable (see, for example, [7]). Given a putative integrable Hamiltonian, what might the invariants look like? This paper tells us that for S_n invariant systems *not* of Calogero–Moser type we should look for conserved quantities quartic and above in the momenta.

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